

On convergence of the distributions of statistics with random sample sizes to normal variance-mean mixtures

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Abstract

We prove a general transfer theorem for multivariate random sequences with independent random indexes in the double array limit setting. We also prove its partial inverse providing necessary and sufficient conditions for the convergence of randomly indexed random sequences. Special attention is paid to the case where the elements of the basic double array are formed as statistics constructed from samples with random sizes. Under rather natural conditions we prove the theorem on convergence of the distributions of such statistics to multivariate normal variance-mean mixtures and, in particular, to multivariate generalized hyperbolic laws.

Keywords: random sequence, random index, transfer theorem, samples with random sizes, normal variance-mean mixture

2010 MSC: 60F05, 60G50, 62E20

1. Introduction

In classical problems of mathematical statistics, the size of the available sample, i. e., the number of available observations, is traditionally assumed to be deterministic. In the asymptotic settings it plays the role of infinitely increasing *known* parameter. At the same time, in practice very often the data to be analyzed is collected or registered during a certain period of time and the flow of informative events each of which brings a next observation forms a random point process, so that the number of available observations is unknown till the end of the process of their registration and also must be

treated as a (random) observation. For example, this is so in insurance statistics where during different accounting periods different numbers of insurance events (insurance claims and/or insurance contracts) occur; in medical statistics where the number of patients with a certain disease varies from month to month due to seasonal factors or from year to year due to some epidemic reasons; in quality control where the number of failed items differs from lot to lot; in high-frequency financial statistics where the number of events in a limit order book during a time unit essentially depends on the intensity of order flows, etc. In these cases the number of available observations as well as the observations themselves are unknown beforehand and should be treated as random. Therefore it is quite reasonable to study the asymptotic behavior of general statistics constructed from samples with random sizes for the purpose of construction of suitable and reasonable asymptotic approximations. As this is so, to obtain non-trivial asymptotic distributions in limit theorems of probability theory and mathematical statistics, an appropriate centering and normalization of random variables and vectors under consideration must be used. It should be especially noted that to obtain reasonable approximation to the distribution of the basic statistics, both centering and normalizing values should be non-random. Otherwise the approximate distribution becomes random itself and, for example, the problem of evaluation of quantiles or significance levels becomes senseless.

In asymptotic settings, statistics constructed from samples with random sizes are special cases of random sequences with random indices. The randomness of indices usually leads to that the limit distributions for the corresponding random sequences are heavy-tailed even in the situations where the distributions of non-randomly indexed random sequences are asymptotically normal see, e. g., [7, 8, 17]. For example, if a statistic which is asymptotically normal in the traditional sense, is constructed on the basis of a sample with random size having negative binomial distribution, then instead of the expected normal law, the Student distribution appears as an asymptotic law for this statistic.

The literature on random sequences with random indexes is extensive, see, e. g., the references above and the references therein.

Although the mathematical theory of random sequences with random indexes is well-developed, there still remain some unsolved problems. For example, convenient conditions for the convergence of the distributions of general statistics constructed from samples with random sizes to normal variance-mean mixtures have not been found yet. The desire to fill this theo-

retical gap is quite natural. Another motivation for this research is practical and is as follows. In applied probability there is a convention, apparently going back to the book [16], according to which a model distribution is reasonable and/or justified enough only if it is an *asymptotic approximation*, that is, there exist a more or less simple limit setting and the corresponding limit theorem in which the model under consideration is a limit distribution. The existence of such a setting may bring a deeper insight into the phenomena under consideration than just fitting a more or less convenient model.

General normal variance-mean mixtures are examples of such convenient models widely used to describe observed statistical regularities in many fields. In particular, in 1977–78 O. Barndorff-Nielsen [1], [2] introduced the class of *generalized hyperbolic distributions* as a class of special univariate variance-mean mixtures of normal laws in which the mixing is carried out in one parameter since location and scale parameters of the mixed normal distribution are directly linked. The range of applications of generalized hyperbolic distributions varies from the theory of turbulence or particle size description to financial mathematics, see [4]. Multivariate generalized hyperbolic distributions were introduced in the seminal paper [1] mentioned above as a natural generalization of the univariate case. They were further investigated in [10] and [11]. It is a convention to explain such a good adequacy of generalized hyperbolic models by that they possess many parameters to be suitably adjusted. But actually, it would be considerably more reasonable to explain this phenomenon by limit theorems yielding the possibility of the use of generalized hyperbolic distributions as convenient *asymptotic approximations*.

The main results presented in this paper deal with the description of conditions which provide the convergence of the distributions of statistics constructed from samples with random sizes to multivariate normal variance-mean mixtures, in particular, to multivariate generalized hyperbolic laws. The conditions presented below are formulated in terms of the asymptotic behavior of random sample sizes and have the ‘if and only if’ form. This circumstance proved to be very promising and constructive. For example, in [24] a problem of construction of suitable approximations to the distribution of the so-called order flow imbalance process in high-frequency trading systems was considered. It was empirically shown that generalized hyperbolic distributions are very likely models for that. But these distributions are variance-mean mixtures with *one* mixing parameter. By means of one-

dimensional limit theorems for random sums, this fact directly lead to theoretical understanding that the intensities of buy and sell orders actually must be proportional to *one and the same* random process reflecting general market agitation. This theoretical inference concerning the flows intensities then found its statistical proof, see [24]. In other words, the ‘if and only if’ character of the presented conditions makes testing goodness-of-fit of financial data with generalized hyperbolic models *equivalent* to testing goodness-of-fit of the corresponding flow intensities (i. e., volatilities) with generalized inverse Gaussian models, which is much simpler.

In the present paper both structural and multivariate generalizations of some results of [24] to general statistics are presented. The paper is organized as follows. Basic notation is introduced in Section 2. Here an auxiliary result on the asymptotic rapprochement of the distributions of randomly indexed random sequences with special scale-location mixtures is proved. In Section 3 of the present paper we prove a general transfer theorem for random sequences with independent random indexes in the double array limit setting. We also prove its partial inverse providing necessary and sufficient conditions for the convergence of randomly indexed random indexes. Following the lines of [20], we first formulate a general result improving some results of [20, 7] by removing some superfluous assumptions and relaxing some conditions. Special attention is paid to the case where the elements of the basic double array are formed as statistics constructed from samples with random sizes. This case is considered in Section 4 where a theorem establishing necessary and sufficient conditions of convergence to general multivariate variance-mean mixtures is proved. As a corollary, in Section 5 we deduce a criterion of convergence of the distributions of statistics constructed from samples with random sizes to multivariate generalized hyperbolic distributions.

2. Notation. Auxiliary results

Let us introduce basic notations to be used throughout this paper. Let $m \in \mathbb{N}$. The vectors $\mathbf{x} = (x^{(1)}, \dots, x^{(m)})^\top$ are elements of \mathbb{R}^m , the superscript $^\top$ stands for the transpose of a vector or matrix. The scalar product in \mathbb{R}^m will be denoted $\langle \cdot, \cdot \rangle$: $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{y} = x^{(1)}y^{(1)} + \dots + x^{(m)}y^{(m)}$. As usual, the Euclidean norm of \mathbf{x} is $\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}$. If A is a real-valued $(m \times m)$ -square matrix, then $\det(A)$ denotes the determinant of A . The $(m \times m)$ -identity matrix is denoted \mathbf{I} . To properly distinguish between the real number zero and the zero vector, we write $0 \in \mathbb{R}$ and $\mathbf{0} = (0, \dots, 0)^\top \in \mathbb{R}^m$. The notation

$N_{\mathbf{a}, \Sigma}$ will be used for the m -dimensional normal distribution with mean vector \mathbf{a} and covariance matrix Σ . The distribution function of the one-dimensional standard normal distribution will be denoted $\Phi(x)$,

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy, \quad x \in \mathbb{R}.$$

Assume that all the random variables and vectors considered in this paper are defined on one and the same probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The symbols \mathfrak{B}_m and \mathfrak{B}_+ will denote the Borel sigma-algebras of subsets of \mathbb{R}_m and $\mathbb{R}_+ \equiv [0, \infty)$, respectively. In what follows the symbols $\stackrel{d}{=}$ and \implies will denote coincidence of distributions and weak convergence (convergence in distribution). We will write $\mathcal{L}(\mathbf{X})$ to denote the distribution of a random vector \mathbf{X} . A family $\{\mathbf{X}_j\}_{j \in \mathbb{N}}$ of \mathbb{R}^m -valued random vectors is said to be *weakly relatively compact*, if each sequence of its elements contains a weakly convergent subsequence. In the finite-dimensional case the weak relative compactness of a family $\{\mathbf{X}_j\}_{j \in \mathbb{N}}$ is equivalent to its *tightness* $\lim_{R \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P}(\|\mathbf{X}_n\| > R) = 0$ (see, e. g., [26]).

Let $\{\mathbf{S}_{n,k} = (S_{n,k}^{(1)}, \dots, S_{n,k}^{(m)})^\top\}$, $n, k \in \mathbb{N}$, be a double array of \mathbb{R}^m -valued random vectors. For $n, k \in \mathbb{N}$ let $\mathbf{a}_{n,k} = (a_{n,k}^{(1)}, \dots, a_{n,k}^{(m)})^\top \in \mathbb{R}^m$ be non-random vectors and $b_{n,k} \in \mathbb{R}$ be real numbers such that $b_{n,k} > 0$. The purpose of the vectors $\mathbf{a}_{n,k}$ and numbers $b_{n,k}$ is to provide weak relative compactness of the family of the random vectors $\{\mathbf{Y}_{n,k} \equiv b_{n,k}^{-1}(\mathbf{S}_{n,k} - \mathbf{a}_{n,k})\}_{n,k \in \mathbb{N}}$ in the cases where it is required.

Consider a family $\{N_n\}_{n \in \mathbb{N}}$ of nonnegative integer random variables such that for each $n, k \in \mathbb{N}$ the random variables N_n and random vectors $\mathbf{S}_{n,k}$ are independent. Especially note that we do not assume the row-wise independence of $\{\mathbf{S}_{n,k}\}_{k \geq 1}$. Let $\mathbf{c}_n = (c_n^{(1)}, \dots, c_n^{(m)})^\top \in \mathbb{R}^m$ be non-random vectors and d_n be real numbers, $n \in \mathbb{N}$, such that $d_n > 0$. Our aim is to study the asymptotic behavior of the random vectors $\mathbf{Z}_n \equiv d_n^{-1}(\mathbf{S}_{n,N_n} - \mathbf{c}_n)$ as $n \rightarrow \infty$ and find rather simple conditions under which the limit laws for \mathbf{Z}_n have the form of normal variance-mean mixtures. In order to do so we first formulate a somewhat more general result following the lines of [20], removing superfluous assumptions, relaxing the conditions and generalizing the results of that paper.

The characteristic functions of the random vectors $\mathbf{Y}_{n,k}$ and \mathbf{Z}_n will be denoted $h_{n,k}(\mathbf{t})$ and $f_n(\mathbf{t})$, respectively, $\mathbf{t} \in \mathbb{R}^m$. Let \mathbf{Y} be an \mathbb{R}^m -valued random vector whose characteristic function will be denoted $h(\mathbf{t})$,

$\mathbf{t} \in \mathbb{R}^m$. Introduce the random variables $U_n = d_n^{-1}b_{n,N_n}$. Let $\mathbf{V}_n = (V_n^{(1)}, \dots, V_n^{(m)})^\top$ where $V_n^{(k)} = d_n^{-1}(a_{n,N_n}^{(k)} - c_n^{(k)})$ is the k th component of the random vector $d_n^{-1}(\mathbf{a}_{n,N_n} - \mathbf{c}_n)$. In what follows by \mathbf{W}_n we will denote the $(m+1)$ -dimensional compound random vector $\mathbf{W}_n = (U_n, \mathbf{V}_n^\top)^\top = (U_n, V_n^{(1)}, \dots, V_n^{(m)})^\top$.

For $\mathbf{t} \in \mathbb{R}^m$ consider the function

$$g_n(\mathbf{t}) \equiv \mathbb{E} h(U_n \mathbf{t}) e^{i\langle \mathbf{t}, \mathbf{V}_n \rangle} = \sum_{k=1}^{\infty} \mathbb{P}(N_n = k) e^{i\langle \mathbf{t}, d_n^{-1}(\mathbf{a}_{n,k} - \mathbf{c}_n) \rangle} h(d_n^{-1}b_{n,k} \mathbf{t}). \quad (1)$$

It can be easily seen that $g_n(\mathbf{t})$ is the characteristic function of the random vector $U_n \cdot \mathbf{Y} + \mathbf{V}_n$ where the random vector \mathbf{Y} is independent of the random vector \mathbf{W}_n .

In the double-array limit setting considered in this paper, to obtain non-trivial limit laws for \mathbf{Z}_n we require the following additional *coherency condition*: for any $T \in (0, \infty)$

$$\lim_{n \rightarrow \infty} \mathbb{E} \sup_{\|\mathbf{t}\| \leq T} |h_{n,N_n}(\mathbf{t}) - h(\mathbf{t})| = 0. \quad (2)$$

REMARK 1. It can be easily verified that, since the values under the expectation sign in (2) are nonnegative and bounded (by two), then coherency condition (2) is equivalent to that $\sup_{\|\mathbf{t}\| \leq T} |h_{n,N_n}(\mathbf{t}) - h(\mathbf{t})| \rightarrow 0$ in probability as $n \rightarrow \infty$.

A particular form of the coherency condition depends on the structure of the statistic $\mathbf{S}_{n,k}$. For example, if $\mathbf{S}_{n,k}$ is a sum of independent random variables and $h(\mathbf{t})$ is the standard normal characteristic function, then, as it was demonstrated in [19], coherency condition (2) turns into the easily verifiable *random Lindeberg condition*, whereas the latter is not only sufficient, but necessary as well for the convergence of the distributions of random sums of independent not necessarily identically distributed random variables, see, e. g., [25].

LEMMA 1. *Let the family of random variables $\{U_n\}_{n \in \mathbb{N}}$ be weakly relatively compact. Assume that coherency condition (2) holds. Then for any $\mathbf{t} \in \mathbb{R}^m$ we have*

$$\lim_{n \rightarrow \infty} |f_n(\mathbf{t}) - g_n(\mathbf{t})| = 0. \quad (3)$$

PROOF. Let $\gamma \in (0, \infty)$ be a real number to be specified later. Denote $K_{1,n} \equiv K_{1,n}(\gamma) = \{k : b_{n,k} \leq \gamma d_n\}$, $K_{2,n} \equiv K_{2,n}(\gamma) = \{k : b_{n,k} > \gamma d_n\}$. If

$\mathbf{t} = 0$, then the assertion of the lemma is trivial. Fix an arbitrary $\mathbf{t} \neq 0$. By the formula of total probability we have

$$\begin{aligned}
& |f_n(\mathbf{t}) - g_n(\mathbf{t})| = \\
& = \left| \sum_{k=1}^{\infty} \mathbf{P}(N_n = k) e^{i\langle \mathbf{t}, d_n^{-1}(\mathbf{a}_{n,k} - \mathbf{c}_n) \rangle} [h_{n,k}(d_n^{-1}b_{n,k}\mathbf{t}) - h(d_n^{-1}b_{n,k}\mathbf{t})] \right| \leq \\
& \leq \sum_{k \in K_{1,n}} \mathbf{P}(N_n = k) |h_{n,k}(d_n^{-1}b_{n,k}\mathbf{t}) - h(d_n^{-1}b_{n,k}\mathbf{t})| + \\
& + \sum_{k \in K_{2,n}} \mathbf{P}(N_n = k) |h_{n,k}(d_n^{-1}b_{n,k}\mathbf{t}) - h(d_n^{-1}b_{n,k}\mathbf{t})| \equiv I_1 + I_2. \quad (4)
\end{aligned}$$

Choose an arbitrary $\epsilon > 0$.

First consider I_2 . We obviously have

$$I_2 \leq 2 \sum_{k \in K_{2,n}(\gamma)} \mathbf{P}(N_n = k) = 2\mathbf{P}(U_n > \gamma). \quad (5)$$

The weak relative compactness of the family $\{U_n\}_{n \in \mathbb{N}}$ implies the existence of a $\gamma_1 = \gamma_1(\epsilon)$ such that $\sup_n \mathbf{P}(U_n > \gamma_1) < \epsilon$. Therefore, setting $\gamma = \gamma_1$ from (5) we obtain

$$I_2 < \epsilon. \quad (6)$$

Now consider I_1 with γ chosen above. If $k \in K_{1,n}(\gamma)$, then $\|d_n^{-1}b_{n,k}\mathbf{t}\| \leq \gamma\|\mathbf{t}\|$ and we have

$$\begin{aligned}
I_1 & \leq \sum_{k \in K_{1,n}(\gamma)} \mathbf{P}(N_n = k) \sup_{\|\mathbf{x}\| \leq \gamma\|\mathbf{t}\|} |h_{n,k}(\mathbf{x}) - h(\mathbf{x})| \leq \\
& \leq \mathbf{E} \sup_{\|\mathbf{x}\| \leq \gamma\|\mathbf{t}\|} |f_{n,N_n}(\mathbf{x}) - h(\mathbf{x})|.
\end{aligned}$$

Therefore, coherency condition (2) implies that there exists a number $n_0 = n_0(\epsilon, \gamma)$ such that for all $n \geq n_0$

$$I_1 < \epsilon. \quad (7)$$

Unifying (4), (6) and (7) we obtain that $|f_n(\mathbf{t}) - g_n(\mathbf{t})| < 2\epsilon$ for $n \geq n_0$. The arbitrariness of ϵ proves (3). The lemma is proved.

Lemma 1 makes it possible to use the distribution defined by the characteristic function $g_n(\mathbf{t})$ (see (1)) as an *accompanying asymptotic* approximation to the distribution of the random vector \mathbf{Z}_n . In order to obtain a *limit* approximation, in the next section we formulate and prove the transfer theorem.

3. General transfer theorem and its inversion. The structure of limit laws

THEOREM 1. *Assume that coherency condition (2) holds. If there exist a random variable U and an m -dimensional random vector \mathbf{V} such that the distributions of the $(m+1)$ -dimensional random vectors \mathbf{W}_n converge to that of the random vector $\mathbf{W} = (U, \mathbf{V}^\top)^\top$:*

$$\mathbf{W}_n \Longrightarrow \mathbf{W} \quad (n \rightarrow \infty), \quad (8)$$

then

$$\mathbf{Z}_n \Longrightarrow \mathbf{Z} \stackrel{d}{=} U \cdot \mathbf{Y} + \mathbf{V} \quad (n \rightarrow \infty). \quad (9)$$

where the random vectors \mathbf{Y} and $\mathbf{W} = (U, \mathbf{V}^\top)^\top$ are independent.

PROOF. Treating $\mathbf{t} \in \mathbb{R}^m$ as a fixed parameter, represent the function $g_n(\mathbf{t})$ as $g_n(\mathbf{t}) = \mathbf{E}h(U_n \mathbf{t}) \exp\{i\langle \mathbf{t}, \mathbf{V}_n \rangle\} \equiv \mathbf{E}\varphi_{\mathbf{t}}(\mathbf{W}_n)$. Since for each $\mathbf{t} \in \mathbb{R}^m$ the function $\varphi_{\mathbf{t}}(\mathbf{w}) \equiv h(u\mathbf{t}) \exp\{i\langle \mathbf{t}, \mathbf{v} \rangle\}$, $\mathbf{w} = (u, \mathbf{v}^\top)^\top \in \mathbb{R}^{m+1}$, is bounded and continuous in \mathbf{w} , then by the definition of the weak convergence we have

$$\lim_{n \rightarrow \infty} \mathbf{E}\varphi_{\mathbf{t}}(\mathbf{W}_n) = \mathbf{E}\varphi_{\mathbf{t}}(\mathbf{W}). \quad (10)$$

Using the Fubini theorem it can be easily verified that the function on the right-hand side of (10) is the characteristic function of the random variable $U \cdot \mathbf{Y} + \mathbf{V}$ where the m -dimensional random vector \mathbf{Y} is independent of the $(m+1)$ -dimensional random vector \mathbf{W} . Now the statement of the theorem follows from lemma 1 by the triangle inequality. The theorem is proved.

It is easy to see that relation (9) is equivalent to that the limit law for normalized randomly indexed random vectors \mathbf{Z}_n is a scale-location mixture of the distributions which are limiting for normalized non-randomly indexed random vectors $\mathbf{Y}_{n,k}$. Among all scale-location mixtures, *variance-mean mixtures* attract a special interest. To be more precise, we should speak of *normal variance-mean mixtures* which are defined in the following way.

An \mathbb{R}^m -valued random vector \mathbf{X} is said to have a multivariate normal mean-variance mixture distribution if $\mathbf{X} \stackrel{d}{=} \mathbf{a} + U\mathbf{b} + \sqrt{U}A\mathbf{Y}$, where $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$, A is a real $(m \times m)$ -matrix such that the matrix $\Sigma \equiv AA^\top$ is positive definite, \mathbf{Y} is a random vector with the standard normal distribution $N_{0,I}$ and U is a real-valued, non-negative random variable independent of \mathbf{Y} . Equivalently, a probability measure F on $(\mathbb{R}^m, \mathfrak{B}_m)$ is said to be a multivariate normal mean-variance mixture if

$$F(d\mathbf{x}) = \int_0^\infty N_{\mathbf{b} + z\mathbf{a}, z\Sigma}(d\mathbf{x})G(dz),$$

where the mixing distribution G is a probability measure on $(\mathbb{R}_+, \mathfrak{B}_+)$. In this case we will sometimes briefly write $F = N_{\mathbf{b}+z\mathbf{a}, z\Sigma} \circ G$.

Let us see how these mixtures can appear in the double-array setting under consideration. Assume that the centering vectors $\mathbf{a}_{n,k}$ and \mathbf{c}_n are in some sense proportional to the scaling constants $b_{n,k}$ and d_n . Namely, assume that there exist vectors $\mathbf{a}_n \in \mathbb{R}^m$ and $\mathbf{b}_n \in \mathbb{R}^m$ such that for all $n, k \in \mathbb{N}$ we have $\mathbf{a}_{n,k} = d_n^{-1} b_{n,k}^2 \mathbf{a}_n$, $\mathbf{c}_n = d_n \mathbf{b}_n$, and there exist finite limits $\mathbf{a} = \lim_{n \rightarrow \infty} \mathbf{a}_n$, $\mathbf{b} = \lim_{n \rightarrow \infty} \mathbf{b}_n$. Then under condition (8) $\mathbf{W}_n = (U_n, (U_n^2 \mathbf{a}_n + \mathbf{b}_n)^\top)^\top \implies (U, (U^2 \mathbf{a} + \mathbf{b})^\top)^\top$ ($n \rightarrow \infty$), so that if in theorem 2 \mathbf{Y} has the m -dimensional normal distribution $N_{\mathbf{0}, \Sigma}$, then the limit law for \mathbf{Z}_n takes the form of the normal variance-mean mixture $N_{\mathbf{b}+z\mathbf{a}, z\Sigma} \circ G$ with G being the distribution of U^2 .

In order to prove a result that is a partial inversion of theorem 1, for fixed random vectors \mathbf{Z} and \mathbf{Y} with the characteristic functions $f(\mathbf{t})$ and $h(\mathbf{t})$ introduce the set $\mathcal{W}(\mathbf{Z}|\mathbf{Y})$ containing all $(m+1)$ -dimensional random vectors $\mathbf{W} = (U, \mathbf{V}^\top)^\top$ with $U \in \mathbb{R}$ and $\mathbf{V} \in \mathbb{R}^m$ such that the characteristic function $f(\mathbf{t})$ can be represented as

$$f(\mathbf{t}) = \mathbb{E} h(U\mathbf{t}) e^{i\langle \mathbf{t}, \mathbf{V} \rangle}, \quad \mathbf{t} \in \mathbb{R}^m, \quad (12)$$

and $\mathbb{P}(U \geq 0) = 1$. Whatever random vectors \mathbf{Z} and \mathbf{Y} are, the set $\mathcal{W}(\mathbf{Z}|\mathbf{Y})$ is always nonempty since it trivially contains the vector $(0, \mathbf{Z}^\top)^\top$. It is easy to see that representation (12) is equivalent to that $\mathbf{Z} \stackrel{d}{=} U\mathbf{Y} + \mathbf{V}$.

The set $\mathcal{W}(\mathbf{Z}|\mathbf{Y})$ may contain more than one element. For example, if \mathbf{Y} is the random vector with standard normal distribution $N_{\mathbf{0}, I}$ and $\mathbf{Z} \stackrel{d}{=} \mathbf{T}_1 - \mathbf{T}_2$ where \mathbf{T}_1 and \mathbf{T}_2 are independent random vectors with independent components having the same standard exponential distribution, then along with the vector $(0, (\mathbf{T}_1 - \mathbf{T}_2)^\top)^\top$ the set $\mathcal{W}(\mathbf{Z}|\mathbf{Y})$ contains the vector $(\sqrt{U}, \mathbf{0}^\top)^\top$ where U is a random variable with the standard exponential distribution. In this case \mathbf{Z} has the spherically symmetric Laplace distribution.

Let $\Lambda(\mathbf{X}_1, \mathbf{X}_2)$ be any probability metric which metrizes weak convergence in the space of $(m+1)$ -dimensional random vectors. An example of such a metric is the Lévy–Prokhorov metric (see, e. g., [9] or [30]).

THEOREM 2. *Let the family of random variables $\{U_n\}_{n \in \mathbb{N}}$ be weakly relatively compact. Assume that coherency condition (2) holds. Then a random vector \mathbf{Z} such that*

$$\mathbf{Z}_n \implies \mathbf{Z} \quad (n \rightarrow \infty) \quad (13)$$

with some $\mathbf{c}_n \in \mathbb{R}^m$ exists if and only if there exists a weakly relatively compact sequence of random vectors $\mathbf{W}_n^* \equiv (U_n^*, (\mathbf{V}_n^*)^\top)^\top \in \mathcal{W}(\mathbf{Z}|\mathbf{Y})$, $n \in \mathbb{N}$, such that

$$\lim_{n \rightarrow \infty} \Lambda(\mathbf{W}_n^*, \mathbf{W}_n) = 0. \quad (14)$$

PROOF. “Only if” part. Prove that the sequence $\{\mathbf{V}_n\}_{n \in \mathbb{N}}$ is weakly relatively compact. The indicator function of a set A will be denoted $\mathbb{I}(A)$. By the formula of total probability for an arbitrary $R > 0$ we have

$$\begin{aligned} \mathbb{P}(\|\mathbf{V}_n\| > R) &= \sum_{k=1}^{\infty} \mathbb{P}(N_n = k) \mathbb{I}(\|d_n^{-1}(\mathbf{a}_{n,k} - \mathbf{c}_n)\| > R) = \\ &= \sum_{k=1}^{\infty} \mathbb{P}(N_n = k) \mathbb{P}(\|d_n^{-1}(\mathbf{S}_{n,k} - \mathbf{c}_n) - d_n^{-1}b_{n,k} \cdot b_{n,k}^{-1}(\mathbf{S}_{n,k} - \mathbf{a}_{n,k})\| > R) \leq \\ &\leq \mathbb{P}(2\|\mathbf{Z}_n\| > R) + \sum_{k=1}^{\infty} \mathbb{P}(N_n = k) \mathbb{P}(2d_n^{-1}b_{n,k} \cdot \|\mathbf{Y}_{n,k}\| > R) \equiv \\ &\quad I_{1,n}(R) + I_{2,n}(R). \end{aligned}$$

First consider $I_{2,n}(R)$. Using the set $K_{2,n} = K_{2,n}(\gamma)$ introduced in the preceding section, for an arbitrary $\gamma > 0$ we have

$$\begin{aligned} I_{2,n}(R) &= \sum_{k \in K_{2,n}} \mathbb{P}(N_n = k) \mathbb{P}(2\|\mathbf{Y}_{n,k}\| > Rb_{n,k}^{-1}d_n) + \\ &\quad + \sum_{k \notin K_{2,n}} \mathbb{P}(N_n = k) \mathbb{P}(2\|\mathbf{Y}_{n,k}\| > Rb_{n,k}^{-1}d_n) \leq \\ &\leq \sum_{k \in K_{2,n}} \mathbb{P}(N_n = k) \mathbb{P}(2\gamma\|\mathbf{Y}_{n,k}\| > R) + \mathbb{P}(U_n > \gamma) \leq \\ &\leq \mathbb{P}(2\gamma\|\mathbf{Y}_{n,N_n}\| > R) + \mathbb{P}(U_n > \gamma). \end{aligned} \quad (15)$$

Fix an arbitrary $\epsilon > 0$. Choose $\gamma = \gamma(\epsilon)$ so that

$$\mathbb{P}(U_n > \gamma(\epsilon)) < \epsilon \quad (16)$$

for all $n \in \mathbb{N}$. This is possible due to the weak relative compactness of the family $\{U_n\}_{n \in \mathbb{N}}$. Now choose $R' = R'(\epsilon)$ so that

$$\mathbb{P}(2\gamma(\epsilon)\|\mathbf{Y}_{n,N_n}\| > R'(\epsilon)) < \epsilon. \quad (17)$$

This is possible due to the weak relative compactness of the family $\{\mathbf{Y}_{n,N_n}\}_{n \in \mathbb{N}}$ implied by coherency condition (2). Thus, from (15), (16) and (17) we obtain

$$I_{2,n}(R'(\epsilon)) < 2\epsilon \quad (18)$$

for all $n \in \mathbb{N}$. Now consider $I_{1,n}(R)$. From (13) it follows that there exists an $R'' = R''(\epsilon)$ such that

$$I_{1,n}(R''(\epsilon)) < \epsilon \quad (19)$$

for all $n \in \mathbb{N}$. From (18) and (19) it follows that if $R > \max\{R', R''\}$, then $\sup_n \mathbf{P}(\|\mathbf{V}_n\| > R) < 3\epsilon$ and by virtue of the arbitrariness of $\epsilon > 0$, the family $\{\mathbf{V}_n\}_{n \in \mathbb{N}}$ is weakly relatively compact. Hence, the family of random vectors $\{\mathbf{W}_n\}_{n \in \mathbb{N}}$ is weakly relatively compact.

Denote $\epsilon_n = \inf\{\Lambda(\mathbf{W}_n, \mathbf{W}) : \mathbf{W} \in \mathcal{W}(\mathbf{Z}|\mathbf{Y})\}$, $n = 1, 2, \dots$. Prove that $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Assume the contrary. In this case $\epsilon_n \geq M$ for some $M > 0$ and all n from some subsequence \mathcal{N} of natural numbers. Choose a subsequence $\mathcal{N}_1 \subseteq \mathcal{N}$ so that the sequence of random vectors $\{\mathbf{W}_n\}_{n \in \mathcal{N}_1}$ weakly converges to some random vector \mathbf{W} . As this is so, for all $n \in \mathcal{N}_1$ large enough we will have $\Lambda(\mathbf{W}_n, \mathbf{W}) < M$. Applying theorem 1 to the sequence $\{\mathbf{W}_n\}_{n \in \mathcal{N}_1}$ we make sure that $\mathbf{W} \in \mathcal{W}(\mathbf{Z}|\mathbf{Y})$ since condition (13) implies the coincidence of the limits of all convergent subsequences of $\{\mathbf{Z}_n\}$. We arrive at the contradiction with the assumption that $\epsilon_n > M$ for all $n \in \mathcal{N}_1$. Hence, $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$ choose a vector $\mathbf{W}_n^* \in \mathcal{W}(\mathbf{Z}|\mathbf{Y})$ such that $\Lambda(\mathbf{W}_n, \mathbf{W}_n^*) \leq \epsilon_n + \frac{1}{n}$. The sequence $\{\mathbf{W}_n^*\}_{n \in \mathbb{N}}$ obviously satisfies condition (14). Its weak relative compactness follows from (14) and the weak relative compactness of the sequence $\{\mathbf{W}_n\}_{n \in \mathbb{N}}$ established above.

“If” part. Assume that the sequence $\{\mathbf{Z}_n\}_{n \in \mathbb{N}}$ does not converge weakly to \mathbf{Z} as $n \rightarrow \infty$. In that case the inequality $\Lambda((0, \mathbf{Z}_n^\top)^\top, (0, \mathbf{Z}^\top)^\top) \geq M$ holds for some $M > 0$ and all n from some subsequence \mathcal{N} of natural numbers. Choose a subsequence $\mathcal{N}_1 \subseteq \mathcal{N}$ so that the sequence of random vectors $\{\mathbf{W}_n^* = (U_n^*, (\mathbf{V}_n^*)^\top)^\top\}_{n \in \mathcal{N}_1}$ weakly converges to some random vector \mathbf{W} . Repeating the reasoning used to prove theorem 1 we make sure that $\mathbf{E}e^{i\langle t, \mathbf{Z} \rangle} = \mathbf{E}h(U_n^* \mathbf{t})e^{i\langle t, \mathbf{V}_n^* \rangle} \rightarrow \mathbf{E}h(U \mathbf{t})e^{i\langle t, \mathbf{V} \rangle}$ as $n \rightarrow \infty$, $n \in \mathcal{N}_1$, for any $t \in \mathbb{R}$ that is, $\mathbf{W} \in \mathcal{W}(\mathbf{Z}|\mathbf{Y})$. From the triangle inequality $\Lambda(\mathbf{W}_n, \mathbf{W}) \leq \Lambda(\mathbf{W}_n, \mathbf{W}_n^*) + \Lambda(\mathbf{W}_n^*, \mathbf{W})$ and condition (14) it follows that $\Lambda(\mathbf{W}_n, \mathbf{W}) \rightarrow 0$ as $n \rightarrow \infty$, $n \in \mathcal{N}_1$. Apply theorem 1 to the double array $\{\mathbf{Y}_{n,k}\}_{k \in \mathbb{N}, n \in \mathcal{N}_1}$ and the sequence $\{\mathbf{W}_n\}_{n \in \mathcal{N}_1}$. As a result we obtain that $\Lambda((0, \mathbf{Z}_n^\top)^\top, (0, \mathbf{Z}^\top)^\top) \rightarrow 0$ as $n \rightarrow \infty$, $n \in \mathcal{N}_1$, contradicting the assumption that $\Lambda((0, \mathbf{Z}_n^\top)^\top, (0, \mathbf{Z}^\top)^\top) \geq M > 0$ for $n \in \mathcal{N}_1$. Thus, the theorem is completely proved.

REMARK 2. It should be noted that in [20] and some subsequent papers a stronger and less convenient version of the coherency condition was used. Furthermore, in [20] and the subsequent papers the statements analogous to lemma 1 and theorems 1 and 2 were proved under the additional assumption of the weak relative compactness of the family $\{\mathbf{Y}_{n,k}\}_{n,k \in \mathbb{N}}$.

4. Limit theorems for statistics constructed from samples with random sizes

Let $\{\mathbf{X}_{n,j}\}_{j \geq 1}$, $n \in \mathbb{N}$, be a double array of row-wise independent not necessarily identically distributed random vectors with values in \mathbb{R}^r , $r \in \mathbb{N}$. For $n, k \in \mathbb{N}$ let $\mathbf{T}_{n,k} = \mathbf{T}_{n,k}(\mathbf{X}_{n,1}, \dots, \mathbf{X}_{n,k})$ be a statistic, i.e., a measurable function of $\mathbf{X}_{n,1}, \dots, \mathbf{X}_{n,k}$ with values in \mathbb{R}^m . For each $n \geq 1$ we define a random vector \mathbf{T}_{n,N_n} by setting $\mathbf{T}_{n,N_n}(\omega) \equiv \mathbf{T}_{n,N_n(\omega)}(\mathbf{X}_{n,1}(\omega), \dots, \mathbf{X}_{n,N_n(\omega)}(\omega))$, $\omega \in \Omega$.

Let θ_n be \mathbb{R}^m -valued vectors, $n \in \mathbb{N}$. In this section we will assume that the random vectors $\mathbf{S}_{n,k}$ have the form $\mathbf{S}_{n,k} = \mathbf{T}_{n,k} - \theta_n$, $n, k \in \mathbb{N}$. Concerning the normalizing constants and vectors we will assume that there exist m -dimensional vectors \mathbf{a} , \mathbf{a}_n , \mathbf{b} , \mathbf{b}_n and positive numbers σ_n such that

$$\mathbf{a}_n \rightarrow \mathbf{a}, \quad \mathbf{b}_n \rightarrow \mathbf{b} \quad (n \rightarrow \infty) \quad (20)$$

and for all $n, k \in \mathbb{N}$

$$b_{n,k} = (\sigma_n \sqrt{k})^{-1}, \quad d_n = (\sigma_n \sqrt{n})^{-1}, \quad \mathbf{a}_{n,k} = (\sigma_n k)^{-1} \sqrt{n} \mathbf{a}_n, \quad \mathbf{c}_n = (\sigma_n \sqrt{n})^{-1} \mathbf{b}_n \quad (21)$$

so that

$$\mathbf{Y}_{n,k} = \sigma_n \sqrt{k}(\mathbf{T}_{n,k} - \theta_n) - \sqrt{n/k} \mathbf{a}_n \quad \text{and} \quad \mathbf{Z}_n = \sigma_n \sqrt{n}(\mathbf{T}_{n,N_n} - \theta_n) - \mathbf{b}_n.$$

As this is so, $\sigma_n^2 \mathbf{I}$ can be regarded as the asymptotic variance of $\mathbf{T}_{n,k}$ as $k \rightarrow \infty$ whereas the bias of $\mathbf{T}_{n,k}$ is $\sqrt{n}(k\sigma_n)^{-1} \mathbf{a}_n$.

Recall that the characteristic function of the normal distribution in \mathbb{R}^m with zero expectation and covariance matrix Σ is $\varphi(\mathbf{t}) = \exp\{-\frac{1}{2}\mathbf{t}^\top \Sigma \mathbf{t}\}$, $\mathbf{t} \in \mathbb{R}^m$. In what follows we will assume that the statistic $\mathbf{T}_{n,k}$ is asymptotically normal in the following sense: there exists a positive definite symmetric matrix Σ such that for any $T \in (0, \infty)$

$$\lim_{n \rightarrow \infty} \mathbb{E} \sup_{\|\mathbf{t}\| \leq T} |h_{n,N_n}(\mathbf{t}) - \exp\{-\frac{1}{2}\mathbf{t}^\top \Sigma \mathbf{t}\}| = 0, \quad (22)$$

where $h_{n,k}(\mathbf{t})$ is the characteristic function of the random vector $\mathbf{Y}_{n,k}$.

THEOREM 3. *Let the family of random variables $\{n/N_n\}_{n \in \mathbb{N}}$ be weakly relatively compact, the normalizing constants have the form (21) and satisfy condition (20). Assume that the statistic $\mathbf{T}_{n,k}$ is asymptotically normal so that condition (22) holds. Then a random vector \mathbf{Z} such that*

$$\sigma_n \sqrt{n}(\mathbf{T}_{n,N_n} - \theta_n) - \mathbf{b}_n \Longrightarrow \mathbf{Z} \quad (n \rightarrow \infty)$$

exists if and only if there exists a distribution function G such that $G(0) = 0$, the distribution F of \mathbf{Z} has the form $F = N_{\mathbf{b} + z\mathbf{a}, z\Sigma} \circ G$ and

$$P(n/N_n < x) \implies G(x) \quad (n \rightarrow \infty). \quad (23)$$

PROOF. We will deduce theorem 3 as a corollary of theorem 2. First, notice that condition (22) is actually the coherency condition (2) with $h(\mathbf{t}) \equiv \exp\{-\frac{1}{2}\mathbf{t}^\top \Sigma \mathbf{t}\}$.

Second, notice that, obviously, each one-dimensional marginal distribution of a multivariate normal variance-mean mixture is a one-dimensional normal variance-mean mixture. Recently in [23] it was proved that one-dimensional normal variance-mean mixtures are identifiable, that is, if $a \in \mathbb{R}$, $\sigma > 0$, $P(Y < x) \equiv \Phi(x)$ and U_1 and U_2 are two nonnegative random variables, then the identity

$$E\Phi\left(\frac{x - aU_1}{\sigma\sqrt{U_1}}\right) \equiv E\Phi\left(\frac{x - aU_2}{\sigma\sqrt{U_2}}\right)$$

implies that $U_1 \stackrel{d}{=} U_2$. Hence it follows that the set $\mathcal{W}(\mathbf{Z}|\mathbf{Y})$ contains at most one vector of the form $\mathbf{W} = (\sigma\sqrt{U}, (U\mathbf{a} + \mathbf{b})^\top)^\top$. This means that in the case under consideration condition (14) reduces to (23). The theorem is proved.

REMARK 3. Note that the statistics $\mathbf{T}_{n,k}$ in the coherency condition are centered, whereas if $\mathbf{b}_n = \mathbf{b} = \mathbf{0}$ and $\mathbf{a} \neq \mathbf{0}$, then the limit distribution for statistics constructed from samples with random sizes becomes skew unlike in the classical situation, where the presence of the systematic bias of the original statistic results in that the limit distribution becomes just shifted. So, if the limit normal variance-mean mixture is skew, then it can be suspected that the original statistics are actually biased.

The class of normal variance-mean mixtures is very wide. For example, it contains generalized hyperbolic laws with generalized inverse Gaussian mixing distributions, in particular, (a) symmetric and non-symmetric (skew) Student distributions (including Cauchy distribution), to which there correspond inverse gamma mixing distributions; (b) variance gamma (VG) distributions (including symmetric and non-symmetric Laplace distributions), to which there correspond gamma mixing distributions; (c) normal\inverse Gaussian (NIG) distributions to which there correspond inverse Gaussian mixing distributions, and many other types. Along with generalized hyperbolic laws, the class of normal variance-mean mixtures contains symmetric

strictly stable laws with strictly stable mixing distributions concentrated on the positive half-line, generalized exponential power distributions and many other types. By variance-mean mixing many other initially symmetric types represented as pure scale mixtures of normal laws can be skewed, e. g., as it was done to obtain non-symmetric exponential power distributions in [18].

5. Convergence to multivariate generalized hyperbolic distributions

Generalized hyperbolic distributions demonstrate exceptionally high adequacy when they are used to describe statistical regularities in the behavior of characteristics of various complex open systems, in particular, turbulent systems and financial markets. There are dozens of dozens of publications dealing with models based on univariate and multivariate generalized hyperbolic distributions. Just mention the canonic papers [2, 3, 12, 29, 13, 5, 14, 15, 6]. Therefore below we will concentrate our attention on limit theorems establishing the convergence of the distributions of statistics constructed from samples with random sizes to multivariate generalized hyperbolic distributions.

In order to do so we should first recall the definition of the *generalized inverse Gaussian distribution* $GIG_{\nu,\mu,\lambda}$ on \mathfrak{B}_+ . The density of this distribution is denoted $p_{GIG}(x; \nu, \mu, \lambda)$ and has the form

$$p_{GIG}(x; \nu, \mu, \lambda) = \frac{\lambda^{\nu/2}}{2\mu^{\nu/2}K_\nu(\sqrt{\mu\lambda})} \cdot x^{\nu-1} \cdot \exp\left\{-\frac{1}{2}\left(\frac{\mu}{x} + \lambda x\right)\right\}, \quad x > 0.$$

Here $\nu \in \mathbb{R}$,

$$\begin{aligned} \mu &> 0, \quad \lambda \geq 0, \quad \text{if } \nu < 0, \\ \mu &> 0, \quad \lambda > 0, \quad \text{if } \nu = 0, \\ \mu &\geq 0, \quad \lambda > 0, \quad \text{if } \nu > 0, \end{aligned}$$

$K_\nu(z)$ is the modified Bessel function of the third kind with index ν ,

$$K_\nu(z) = \frac{1}{2} \int_0^\infty y^{\nu-1} \exp\left\{-\frac{z}{2}\left(y + \frac{1}{y}\right)\right\} dy, \quad z \in \mathbb{C}, \quad \operatorname{Re} z > 0.$$

According to [27], the generalized inverse Gaussian distribution was introduced in 1946 by Étienne Halphen, who used it to describe monthly volumes of water passing through hydroelectric power stations. In the paper [27] generalized inverse Gaussian distribution was called the *Halphen distribution*. In

1973 this distribution was re-discovered by Herbert Sichel [28], who used it as the mixing law in special mixed Poisson distributions (the *Sichel distributions*, see, e. g., [21]) as discrete distributions with heavy tails. In 1977 these distributions were once more re-discovered by O. Barndorff-Nielsen [1, 2], who, in particular, used them to describe the particle size distribution.

The class of generalized inverse Gaussian distributions is rather rich and contains, in particular, both distributions with exponentially decreasing tails (gamma-distribution ($\mu = 0, \nu > 0$)), and distributions whose tails demonstrate power-type behavior (inverse gamma-distribution ($\lambda = 0, \nu < 0$), inverse Gaussian distribution ($\nu = -\frac{1}{2}$) and its limit case as $\lambda \rightarrow 0$, the Lévy distribution (stable distribution with the characteristic exponent equal to $\frac{1}{2}$ and concentrated on the nonnegative half-line, the distribution of the time for the standard Wiener process to hit the unit level)).

In the final part of his seminal paper [1], O. Barndorff-Nielsen defined the class of multivariate *generalized hyperbolic distributions* as the class of special normal variance-mean mixtures. Namely, let Σ be a positive definite ($m \times m$)-matrix with $\det(\Sigma)=1$, \mathbf{a} and \mathbf{b} be m -dimensional vectors. Then the m -dimensional generalized hyperbolic distribution $GH_{\nu,\mu,\lambda,\mathbf{a},\mathbf{b},\Sigma}$ on \mathfrak{B}_m is defined as

$$GH_{\nu,\mu,\alpha,\mathbf{a},\mathbf{b},\Sigma} = \mathcal{N}_{\mathbf{b}+z\Sigma\mathbf{a}, z\Sigma} \circ GIG(\nu, \mu, \sqrt{\alpha^2 - \langle \mathbf{a}, \Sigma \mathbf{a} \rangle}).$$

Due to the restrictions imposed on the parameters of the generalized inverse Gaussian distribution, the parameters of generalized hyperbolic distribution must fit the conditions $\nu \in \mathbb{R}$, $\alpha, \mu \in \mathbb{R}_+$ and

$$\begin{aligned} \mu &> 0, \quad 0 \leq \langle \mathbf{a}, \Sigma \mathbf{a} \rangle \leq \alpha^2, \quad \text{if } \nu < 0, \\ \mu &> 0, \quad 0 \leq \langle \mathbf{a}, \Sigma \mathbf{a} \rangle < \alpha^2, \quad \text{if } \nu = 0, \\ \mu &\geq 0, \quad 0 \leq \langle \mathbf{a}, \Sigma \mathbf{a} \rangle < \alpha^2, \quad \text{if } \nu > 0, \end{aligned}$$

The corresponding distribution density $p_{GH}(\mathbf{x}; \nu, \mu, \alpha, \mathbf{a}, \mathbf{b}, \Sigma)$ has the form

$$\begin{aligned} p_{GH}(\mathbf{x}; \nu, \mu, \alpha, \mathbf{a}, \mathbf{b}, \Sigma) &= \\ &= \frac{(\alpha^2 - \langle \mathbf{a}, \Sigma \mathbf{a} \rangle)^{\nu/2}}{(2\pi)^{m/2} \alpha^{\nu-m/2} \mu^{\nu/2} K_\nu(\sqrt{\mu(\alpha^2 - \langle \mathbf{a}, \Sigma \mathbf{a} \rangle)})} \sqrt{(\langle \mathbf{x} - \mathbf{b}, \Sigma^{-1}(\mathbf{x} - \mathbf{b}) \rangle + \mu)^{\nu-m/2}} \times \\ &\times K_{\nu-m/2}(\alpha \sqrt{\langle \mathbf{x} - \mathbf{b}, \Sigma^{-1}(\mathbf{x} - \mathbf{b}) \rangle + \mu}) \exp\{\langle \mathbf{a}, \mathbf{x} - \mathbf{b} \rangle\}, \quad \mathbf{x} \in \mathbb{R}^m. \end{aligned}$$

THEOREM 4. *Let the family of random variables $\{n/N_n\}_{n \in \mathbb{N}}$ be weakly relatively compact, the normalizing constants have the form (21) and satisfy condition (20) with some $\mathbf{a}, \mathbf{b} \in \mathbb{R}^m$. Assume that the statistic $\mathbf{T}_{n,k}$ is asymptotically normal so that condition (22) holds with some symmetric positive definite matrix Σ . Then the distribution of a statistic \mathbf{T}_{n,N_n} constructed from the sample with random size N_n weakly converges, as $n \rightarrow \infty$, to an m -dimensional generalized hyperbolic distribution:*

$$\mathcal{L}(\sigma_n \sqrt{n}(\mathbf{T}_{n,N_n} - \theta_n) - \mathbf{b}_n) \Longrightarrow GH_{\nu, \mu, \alpha, \Sigma^{-1} \mathbf{a}, \mathbf{b}, \Sigma}$$

if and only if

$$\mathcal{L}(n^{-1}N_n) \Longrightarrow GIG_{-\nu, \lambda, \mu} \quad (24)$$

with $\lambda = \sqrt{\alpha^2 - \langle \mathbf{a}, \Sigma \mathbf{a} \rangle}$.

This theorem is a straightforward corollary of theorem 3 with the account of a simply verifiable fact that if $\mathcal{L}(\xi) = GIG_{\nu, \mu, \lambda}$, then $\mathcal{L}(\xi^{-1}) = GIG_{-\nu, \lambda, \mu}$.

Theorem 4 can serve as convenient explanation of the high adequacy of generalized hyperbolic Lévy distributions as models of statistical regularities in the behavior of stochastic systems. Moreover, they directly link the mixing distribution in the representation of a generalized hyperbolic distribution with the random sample size which is determined by the intensity of the flow of informative events generating the observations, see, e. g., [24].

According to theorem 4, for example, to obtain the limit multivariate asymmetric Student distribution for \mathbf{T}_{n,N_n} it is necessary and sufficient that in (24) the mixing distribution is the gamma distribution [22]. To obtain the multivariate variance gamma limit distribution for \mathbf{T}_{n,N_n} it is necessary and sufficient that in (24) the mixing distribution is the inverse gamma distribution [22]. In particular, for \mathbf{T}_{n,N_n} to have the limit multivariate asymmetric Laplace distribution it is necessary and sufficient that the limit distribution for $n^{-1}N_n$ is inverse exponential.

Acknowledgement. This research was supported by the Russian Science Foundation (project 14-11-00364).

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